

Monday, October 20, Lecture 22: Laurent series, applications of Residue thm

• Exam next Monday

- Optimization $\leftarrow$ Maybe 1 small question
- Complex Analysis $\leftarrow$ Most of it

Last time...

• Residue thm:

f holomorphic except at $z_0 \in \Omega$.

$$\int_{\gamma} f dz = 2\pi i \operatorname{Res}_{z_0} f$$



• Prime ex:

$\int_{\text{unit circle}}$

$\frac{1}{z} dz = 2\pi i \cdot 1$

$$\text{since } f(z) = \frac{a_{-m}}{(z-z_0)^m} + \dots + \frac{a_{-1}}{z-z_0} + g(z) \\ = \frac{1}{z} \\ \Rightarrow a_{-1} = 1.$$

• Techniques for calculating residue

• Essential singularity = a singularity which is neither a pole nor removable

Ex:

$$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!} \frac{1}{z^2} + \frac{1}{3!} \frac{1}{z^3} + \dots$$

Then (Laurent series)

Let C_1 and C_2 be two circles centered at z_0 .

Let f be holomorphic in the region R between them.

Then f can be expanded as a convergent series in R by

$$f(z) = \sum_{n=0}^{\infty} \frac{a_{-n}}{(z-z_0)^n} + \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

$$= \dots + \frac{a_{-m}}{(z-z_0)^m} + \frac{a_{-m+1}}{(z-z_0)^{m+1}} + \dots + \frac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + \dots + a_m(z-z_0)^m + \dots$$

Principal part

Warning: Expansion depends on C_1, C_2 .



Most important case: the "circles"

$$C_1 = \{z_0\} \text{ and } C_2 = \mathbb{C}.$$

Ex: $z_0 = 0$, so $C_1 = \{0\}$ and $C_2 = \mathbb{C}$

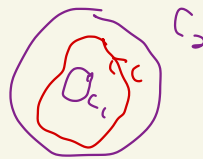
$$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!} \frac{1}{z^2} + \frac{1}{3!} \frac{1}{z^3} + \dots$$



We can get explicit formulae for coefficients by Cauchy's integral formula (exercise): if $C \in R$, and $n \geq 0$,

$$a_n = \int_C \frac{f(z)}{(z-z_0)^{n+1}} dz, \quad a_{-n} = \int_C f(z) (z-z_0)^{-n-1} dz$$

In practice, better to do series expansion.



• Observation: (alternative defn):

• Let f be holomorphic on a nbhd of

z_0 . so,

$$f(z) = \dots \frac{a_{-m}}{(z-z_0)^m} + \dots + \frac{a_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) + \dots$$

Then

• $\text{Res}_{z_0} f = a_{-1}$

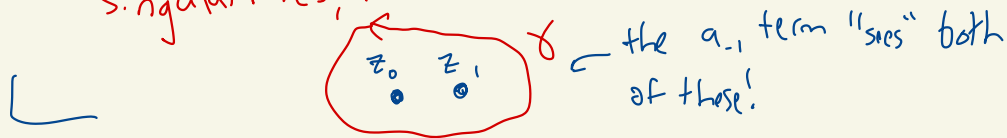
Warning! This is only true for Laurent expansion w.r.t. curves $C_1 = \{z_0\}$ and C_2 small enough so that f is holomorphic between them

• Ex: in upcoming example,

$$\text{Res}_0 f = 6 \quad (\text{from } R_1)$$

but not 2 or 0 (from R_2, R_3 expansions)

• The problem is that if a loop encircles multiple singularities, it "sees" them all and combines them:



• z_0 is an essential singularity iff there are infinitely many a_n with $n < 0$.

• Pole = this Laurent series terminates in the negative direction (like a polynomial in $\frac{1}{z}$)

In practice, it's better to do series expansions. (which again, depend on C_1, C_2)

Ex: find Laurent series for

$$f(z) = \frac{12}{z(2-z)(1+z)}$$

Always partial fractions + geometric series, being careful with convergence.

Soln: 3 singularities, $z=0, 2$, and -1 , all simple poles. converge.

- we should clarify which circles we want, so f is holomorphic in correct region.



$$R_1 = \{z: 0 < |z| < 1\}$$

$$R_2 = \{z: 1 < |z| < 2\}$$

$$R_3 = \{z: |z| > 2\}$$

$$R_1: f(z) = \frac{4}{z} \left(\frac{1}{1+z} + \frac{1}{2-z} \right) = \frac{4}{z} \left(\frac{1}{1-(-z)} + \frac{1}{2} \frac{1}{1-\frac{z}{2}} \right)$$

↑
partial fractions

$$= \frac{4}{z} \left(1 - z + z^2 - z^3 + \dots + \frac{1}{2} + \frac{1}{2} \left(\frac{z}{2} \right) + \frac{1}{2} \left(\frac{z}{2} \right)^2 + \dots \right)$$

Since $C_1 = \{0\}$, $\text{Res}_0 f = 6$.

$$= \frac{6}{z} - 4 + 1 + 4z + \frac{1}{2}z + \dots$$

$$= \frac{6}{z} - 3 + \frac{9}{2}z + \dots$$

Now, for R_2 . Can't do exactly the same trick,
since when $|z| > 1$, series diverges, thanks to $1 - z + z^2 - \dots$

So, we expand

$$\frac{1}{2-z} = \frac{1}{2} \left(1 + \frac{z}{2} + \left(\frac{z}{2}\right)^2 + \dots \right) \leftarrow \text{converges for } |z| < 2$$

and

$$\frac{1}{1+z} = \frac{1}{z} \left(\frac{1}{1+\frac{1}{z}} \right) = \frac{1}{z} \left(1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \dots \right)$$

"contains" 1 pole

converges for $|\frac{1}{z}| < 1 \Leftrightarrow |z| > 1$.

So in R_2 ,

$$f(z) = \frac{4}{z} \left(\frac{1}{1+z} + \frac{1}{2-z} \right)$$

$$= \frac{4}{z} \left(\frac{1}{z} - \frac{1}{z^2} + \frac{1}{z^3} - \dots + \frac{1}{2} + \frac{1}{2} \left(\frac{z}{2} \right) + \frac{1}{2} \left(\frac{z}{2} \right)^2 + \dots \right)$$

Notice:
since inner circle contains multiple poles, a.c. is not residue here

$$= \frac{4}{z^2} - \frac{4}{z^3} + \frac{4}{z^4} + \dots + \frac{2}{z} + 1 + \frac{z}{2} + \dots$$

"contains" 2 poles

And in R_3 , since $|z| > 2$, expand

$$\frac{1}{1+z} = \frac{1}{z} \frac{1}{1+\frac{1}{z}} \quad \frac{1}{2-z} = -\frac{1}{z} \frac{1}{1-\frac{2}{z}}$$

converges since $|\frac{1}{z}| < 1 \Leftrightarrow |z| > 1$

$|\frac{2}{z}| < 1 \Leftrightarrow |z| > 2$

so

$$f(z) = \frac{4}{z} \left(\frac{1}{z} \frac{1}{1+\frac{1}{z}} - \frac{1}{z} \frac{1}{1-\frac{2}{z}} \right) = \frac{4}{z^2} \left(1 - \frac{1}{z} + \frac{1}{z^2} - \dots - \left(1 + \frac{2}{z} + \frac{4}{z^2} + \dots \right) \right)$$

contains 3 poles. $\left[= -\frac{12}{z^3} + \dots \right]$

Main application of Residue Thm: Crucial integrals!

Many integrals in physics and engineering cannot be computed without complex analysis.

Ex: Compute $I = \int_0^{2\pi} \frac{1}{5+4\cos\theta} d\theta$ (these sorts of integrals show up when studying pendulums and other oscillators)

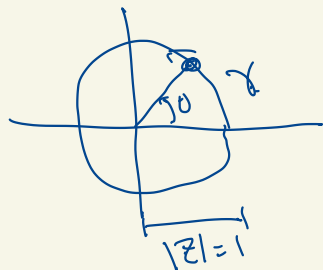
Soln:

Let

$$z = e^{i\theta} \quad (\text{so } z(\theta) = \gamma(\theta) = e^{i\theta}, \theta \in [0, 2\pi])$$

$$dz = ie^{i\theta} d\theta = iz d\theta$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{z + \frac{1}{z}}{2}$$



$$\Rightarrow I = \int_{\gamma} \left(\frac{1}{5+2(z+\frac{1}{z})} \right) \frac{1}{iz} dz$$

$$= \frac{1}{i} \int_{\gamma} \frac{1}{(2z+1)(z+2)} dz$$

Residue Thm: f has poles at $z = -\frac{1}{2}$ and $z = -2$



$$\text{so } iI = 2\pi i \operatorname{Res}_{-\frac{1}{2}} f$$

- Since $-\frac{1}{2}$ is a simple pole,

$$\operatorname{Res}_{-\frac{1}{2}} f = \lim_{z \rightarrow -\frac{1}{2}} (z + \frac{1}{2}) \cdot f = \frac{1}{2(z+2)} \Big|_{z=-\frac{1}{2}} = \frac{1}{3}$$

So

$$I = 2\pi \left(\frac{1}{3} \right) = \frac{2\pi}{3}$$

- Not too hard to see this works for other rational functions of θ !

Ex: Rational functions $\int_{\mathbb{R}} \frac{p(x)}{q(x)} dx$ Hw: Cauchy distribution.